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Problem 1: Find all ordered pairs of positive integers (a,b) such that $\frac{1}{a}+\frac{a}{b}+\frac{1}{ab}=1$.

Note that the equation may be rewritten as

$$b+a^2+1=ab.$$

Isolating b gives
$$b=\frac{a^2+1}{a-1}$$
$$=\frac{a^2-1+2}{a-1}$$
$$=\frac{(a+1)(a-1)}{a-1}+\frac{2}{a-1}$$
$$b=a+1+\frac{2}{a-1}$$

Since a is a positive integer, it is clear that b is also a positive integer iff $\frac{2}{a-1}$ is an integer.

There are only 2 values of a, 2 and 3, such that $\frac{2}{a-1}$ is an integer. In the former case, $b=5$. In the latter, $b=5$. Hence, the only ordered pairs (a,b) satisfying $\frac{1}{a}+\frac{a}{b}+\frac{1}{ab}=1$ over the positive integers are

$$(2,5) \text{ and } (3,5).$$